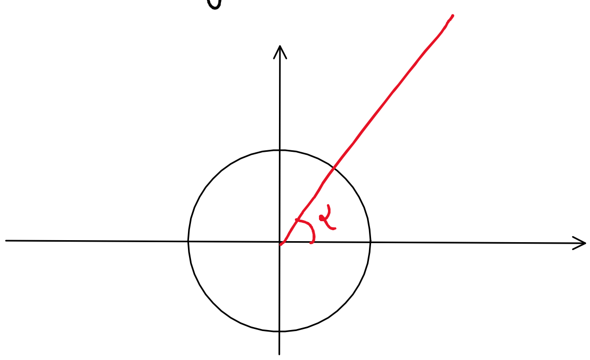
Funzioni trigonometriche (cos / sin / tan)

CIRCONFERENZA GONIOMETRICA : è la circonferenza in $\mathbb{R}^2$ di centro $(0, 0)$ e raggio 1.

Equazione : $x^2 + y^2 = 1$.

$\mathcal{C} := \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$

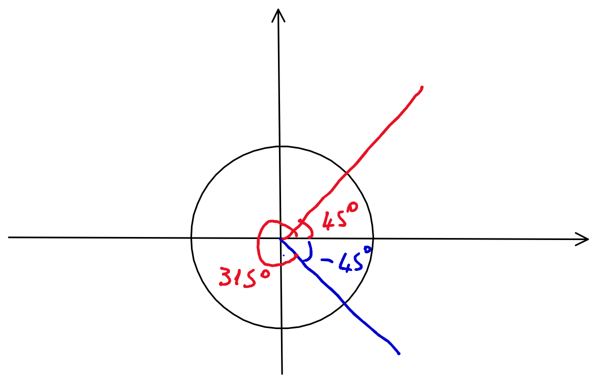
- Sulla circonferenza goniometrica è facile rappresentare gli angoli:



Agli angoli possiamo dare un segno:

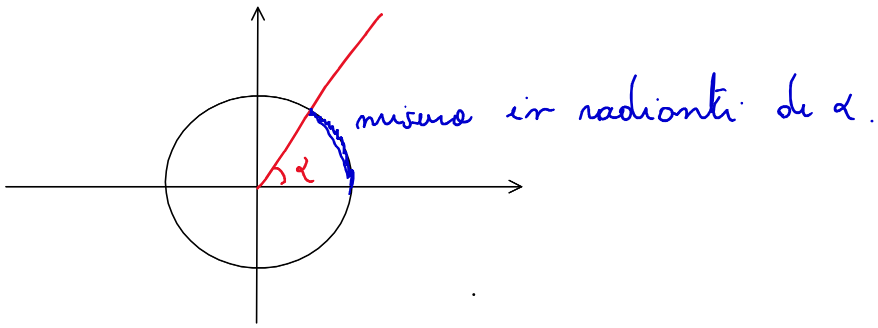
Convenzione :

- Angoli positivi : si ottengono ruotando in senso antiorario
- Angoli negativi : senso orario



- Ogni angolo rappresentato su $\mathcal{C}$ individua un arco.

Def La lunghezza dell'arco su $\mathcal{C}$ che corrisponde a un angolo α è detta MISURA IN RADIANTI DI α.
(lunghezza con segno)



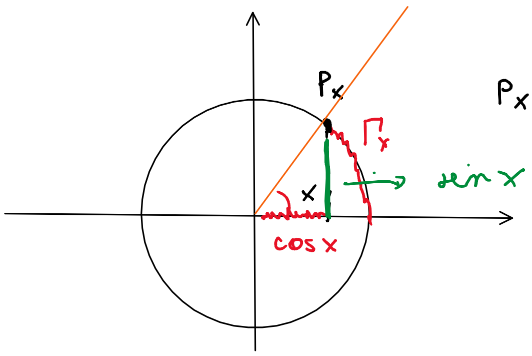
α	radianti
0	0
$180°$	π
$90°$	$\frac{\pi}{2}$
$270°$	$\frac{3}{2}\pi$
$45°$	$\frac{\pi}{4}$
$30°$	$\frac{\pi}{6}$
$60°$	$\frac{\pi}{3}$
$135° = 90° + 45°$	$\frac{\pi}{2}+\frac{\pi}{4} = \pi\left(\frac{1}{2}+\frac{1}{4}\right) = \pi \cdot \frac{3}{4} = \frac{3}{4}\pi$

α : ampiezza in gradi.
x : ampiezza in radianti.
Allora

$$\frac{\alpha}{180} = \frac{x}{\pi} \qquad (\alpha : 180 = x : \pi)$$

$$\left(\alpha = \frac{180}{\pi}x \quad \text{e} \quad x = \frac{\alpha}{180}\cdot\pi\right)$$

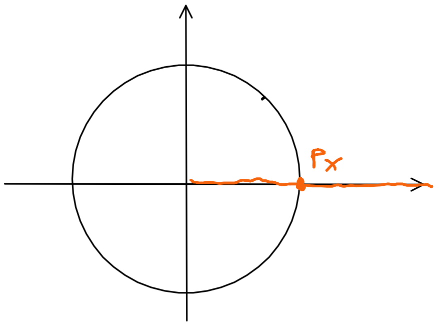
<u>Def</u>: Sia $x \in \mathbb{R}$. Sia Γ_x l'arco sulla circonf. goniometrica (che parte $(1,0)$) che corrisponde a un angolo di ampiezza in radianti x. Sia P_x il punto finale di Γ_x. Le coordinate di P_x si dicono COSENO e SENO di x.



$$P_x = (\cos x, \sin x)$$

Esempi di valori noti di coseno e seno

- $x = 0$

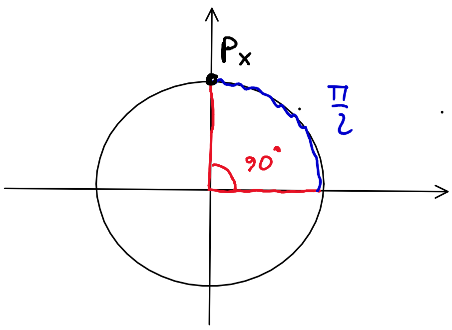


$P_x = (1, 0)$

Quindi $\cos 0 = 1$

$\sin 0 = 0$

- $x = \frac{\pi}{2}$

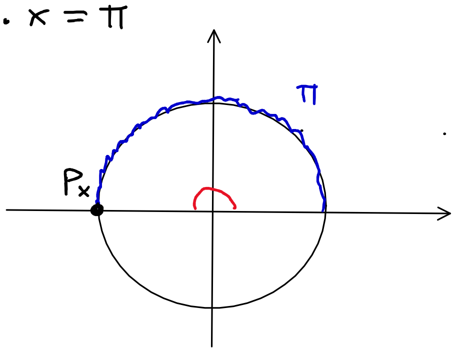


$P_x = (0, 1)$

$\cos \frac{\pi}{2} = 0$

$\sin \frac{\pi}{2} = 1$

- $x = \pi$

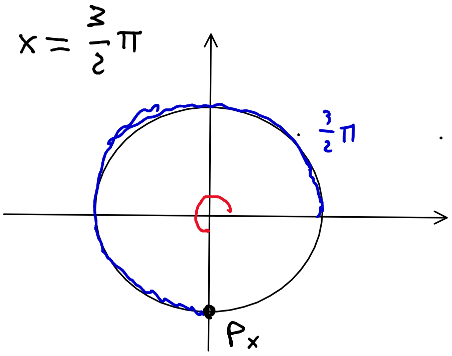


$P_x = (-1, 0)$

$\cos \pi = -1$

$\sin \pi = 0$

- $x = \frac{3}{2}\pi$

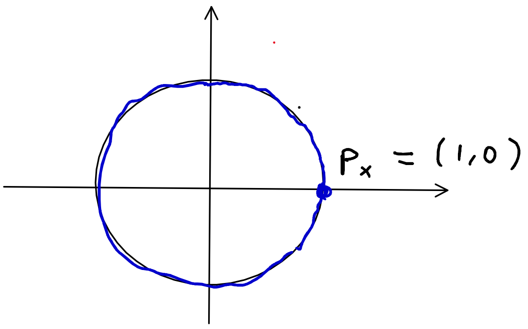


$P_x = (0, -1)$

$\cos \frac{3}{2}\pi = 0$

$\sin \frac{3}{2}\pi = -1$

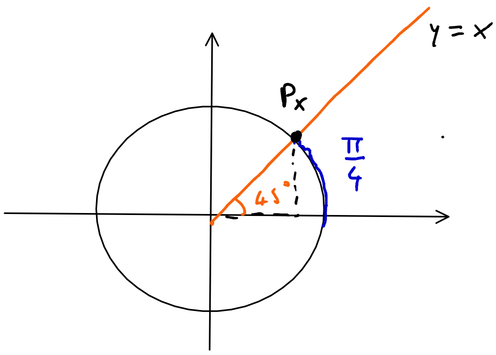
- $x = 2\pi$



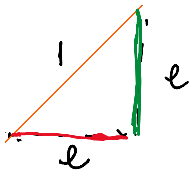
$\cos 2\pi = \cos 0 = 1$

$\sin 2\pi = \sin 0 = 0$

- $x = \frac{\pi}{4}$ $(45°)$



$\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

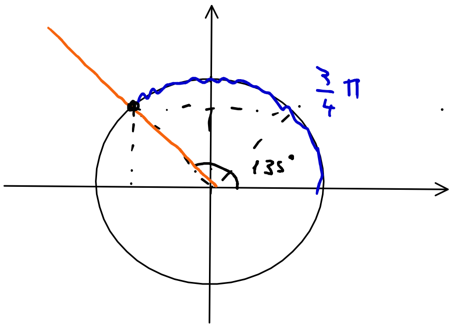


$\ell^2 + \ell^2 = 1$

$2\ell^2 = 1$

$\ell^2 = \frac{1}{2}$ $\ell = \frac{1}{\sqrt{2}}$

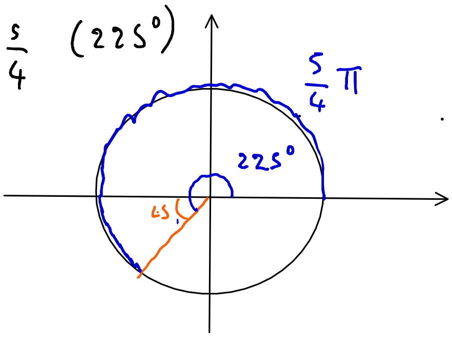
- $x = \frac{3}{4}\pi$ $(135°)$



$\cos \frac{3}{4}\pi = -\frac{1}{\sqrt{2}}$

$\sin \frac{3}{4}\pi = \frac{1}{\sqrt{2}}$

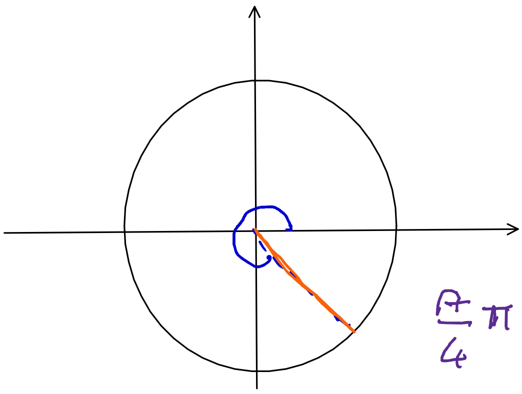
- $x = \frac{5}{4}$ $(225°)$



$\cos \frac{5}{4}\pi = -\frac{1}{\sqrt{2}}$

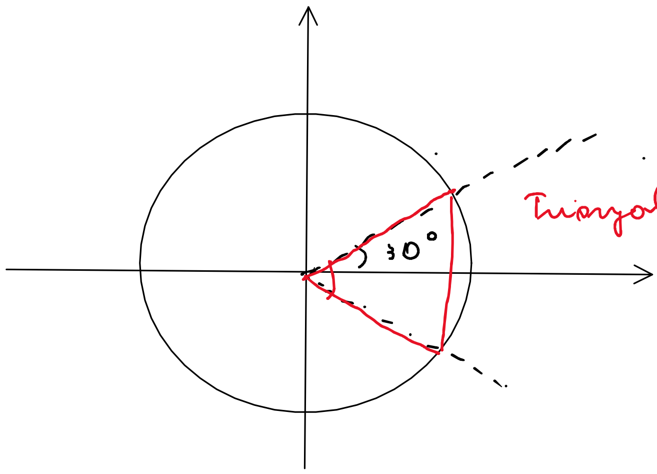
$\sin \frac{5}{4}\pi = -\frac{1}{\sqrt{2}}$

- $\cos \frac{7}{4}\pi = \frac{1}{\sqrt{2}}$ $\qquad$ $\sin \frac{7}{4}\pi = -\frac{1}{\sqrt{2}}$



$\frac{7}{4}\pi$ $\quad \left(0 - -\frac{\pi}{4}\right)$

- $x = \frac{\pi}{6}$ $\quad (30°)$

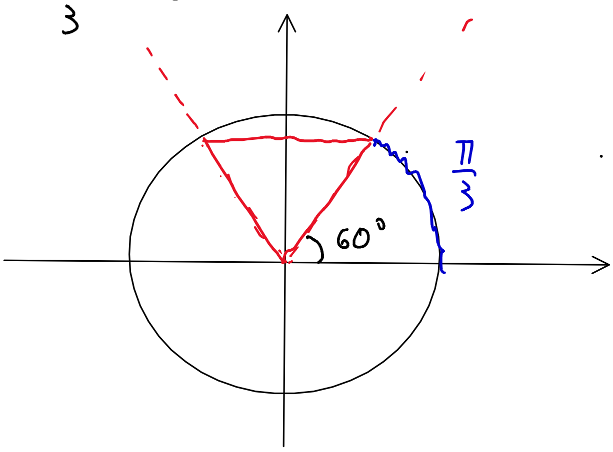


Triangolo equilatero.

$\sin \frac{\pi}{6} = \frac{1}{2}$

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

- $x = \frac{\pi}{3}$ $\quad (60°)$

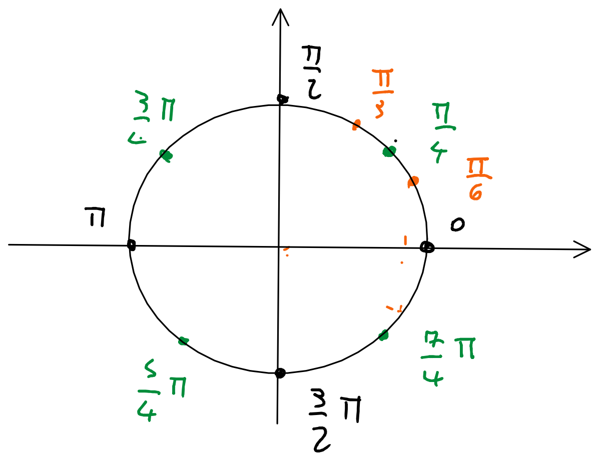


$\cos \frac{\pi}{3} = \frac{1}{2}$

$\sin \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Tabella di riepilogo di valori noti di seno e coseno

x	$\cos x$	$\sin x$
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3}{2}\pi$	0	-1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{3}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{5}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{7}{4}\pi$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

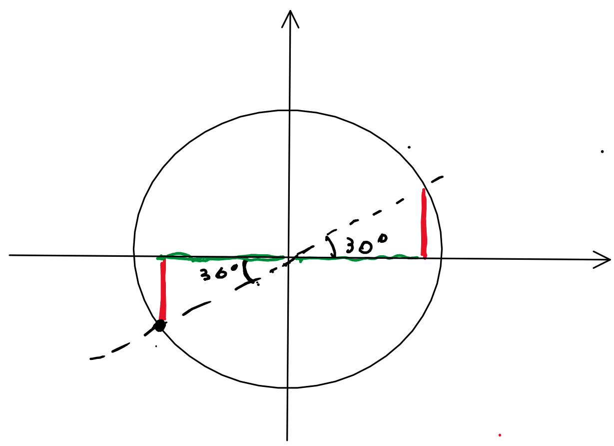


Altri angoli

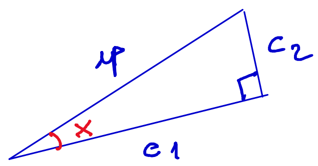
$$x = \frac{7}{6}\pi = \pi + \frac{\pi}{6}$$

$$\cos\left(\frac{7}{6}\pi\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{7}{6}\pi\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$



Interpretazione nei triangoli rettangoli

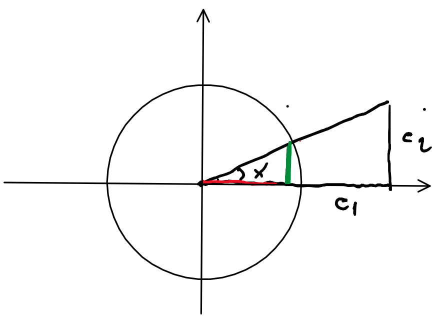


ip : IPOTENUSA (lato opposto all'angolo retto)

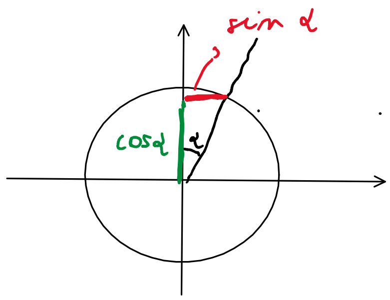
c_1 : CATETO ADIACENTE AD x

c_2 : CATETO OPPOSTO AD x

Si può sempre riflettere e ruotare il triangolo in modo tale che il vertice che corrisponde ad α sia l'origine e il cateto adiacente ad α sia sul semiasse positivo delle x.



$$\frac{ip}{1} = \frac{c_1}{\cos x} = \frac{c_2}{\sin x}$$



Conseguenza:

$c_1 = ip \cdot \cos x$

$c_2 = ip \cdot \sin x$

PROPRIETÀ DI SENO E COSENO

1) $\forall x \in \mathbb{R}$: $\cos^2 x + \sin^2 x = 1$

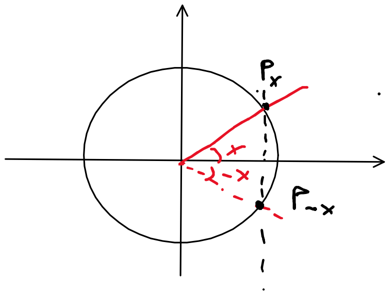
2) $\forall x \in \mathbb{R}$: $|\cos x| \leq 1$, $|\sin x| \leq 1$.

$(-1 \leq \cos x \leq 1 \quad , \quad -1 \leq \sin x \leq 1)$

3) $\forall x \in \mathbb{R}$:

$$\cos(-x) = \cos x$$

$$\sin(-x) = -\sin x$$



4) $\forall x \in \mathbb{R}$:

$$\cos(x + 2\pi) = \cos x$$

$$\sin(x + 2\pi) = \sin x$$

Più in generale :

$$\cos(x + 2k\pi) = \cos x \quad \forall k \in \mathbb{Z}$$

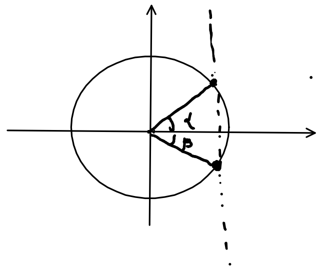
$$\sin(x + 2k\pi) = \sin x$$

5) $\forall \alpha, \beta \in \mathbb{R}$:

$$\cos \alpha = \cos \beta \iff \alpha = \beta + 2k\pi \text{ con } k \in \mathbb{Z}$$

$$\vee$$

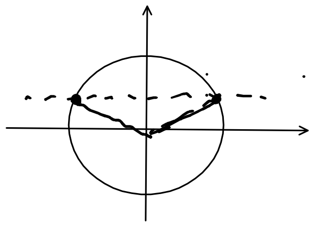
$$\alpha = -\beta + 2k\pi \text{ con } k \in \mathbb{Z}$$



6) $\forall \alpha, \beta \in \mathbb{R}$:

$$\sin \alpha = \sin \beta \iff \alpha = \beta + 2k\pi \text{ con } k \in \mathbb{Z}$$

$$\alpha = \pi - \beta + 2k\pi \text{ con } k \in \mathbb{Z}$$



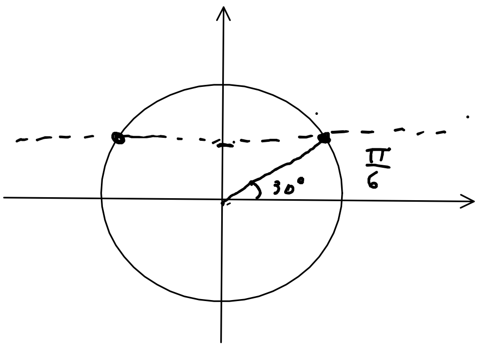
ESEMPIO

Risolvere

$$2 \sin x = 1$$

$$\sin x = \frac{1}{2}$$

$$\sin x = \sin \frac{\pi}{6}$$



$$x = \frac{\pi}{6} + 2k\pi \text{ con } k \in \mathbb{Z} \quad \vee \quad x = \pi - \frac{\pi}{6} + 2k\pi$$

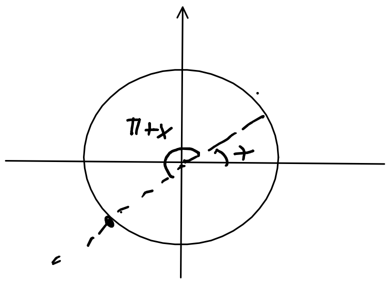
$$x = \frac{5}{6}\pi + 2k\pi$$

FORMULE DI ADDIZIONE E SOTTRAZIONE

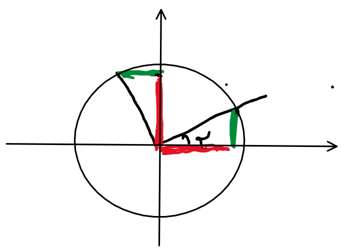
Siano $x, y \in \mathbb{R}$:

1) $\cos(x+y) = \cos x \cos y - \sin x \sin y$
2) $\cos(x-y) = \cos x \cos y + \sin x \sin y$
3) $\sin(x+y) = \sin x \cos y + \cos x \cdot \sin y$
4) $\sin(x-y) = \sin x \cos y - \cos x \sin y$

- $\cos(x+\pi) = -\cos x \quad (\cos x \cdot \underset{-1}{\cos\pi} - \sin x \underset{0}{\sin\pi})$
- $\sin(x+\pi) = -\sin x$



- $\cos(x+\frac{\pi}{2}) = \cos x \cdot \cos\frac{\pi}{2} - \sin x \cdot \sin\frac{\pi}{2}$
  $= 0 - \sin x = -\sin x$
- $\sin(x+\frac{\pi}{2}) = \sin x \cos\frac{\pi}{2} + \cos x \cdot \sin\frac{\pi}{2}$
  $= \cos x$



FORMULE DI DUPLICAZIONE. Sia $x \in \mathbb{R}$. Allora:

1) $\cos(2x) = \cos^2 x - \sin^2 x$
2) $\sin(2x) = 2\sin x \cos x$

OSS Siccome $\cos^2 x + \sin^2 x = 1$, allora

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$

formule alternative per $\cos(2x)$

In particolare

$$\cos(2x) = 2\cos^2 x - 1 \implies \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\implies \cos x = \pm\sqrt{\frac{1+\cos 2x}{2}}$$

$$\cos(2x) = 1 - 2\sin^2 x \implies \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\implies \sin x = \pm\sqrt{\frac{1-\cos 2x}{2}}$$

FORMULE DI BISEZIONE

$\forall \; x \in \mathbb{R}$:

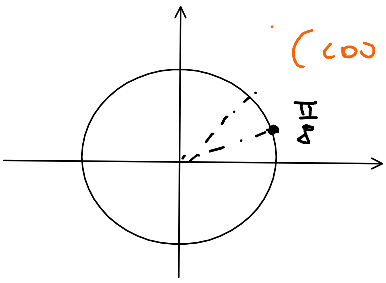
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

Con il segno che va determinato in base alla posizione del punto su C che corrisponde a $\frac{x}{2}$

ESEMPIO

- $$\cos\frac{\pi}{8} = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}$$

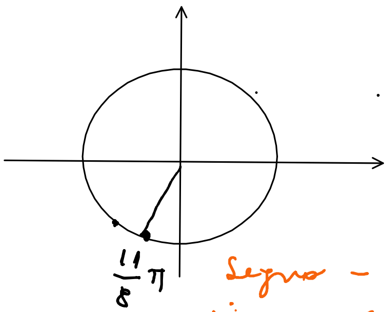


($\cos\frac{\pi}{8} > 0$, $\sin\frac{\pi}{8} > 0$ quindi le formule si applicano con il segno $+$)

e

$$\sin\frac{\pi}{8} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} = +\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$

- $\cos\left(\frac{11}{8}\pi\right)$ $\qquad \frac{11}{8}\pi = \pi + \frac{3}{8}\pi$



Segno $-$ sia per $\cos\frac{11}{8}\pi$ che per $\sin\left(\frac{11}{8}\pi\right)$

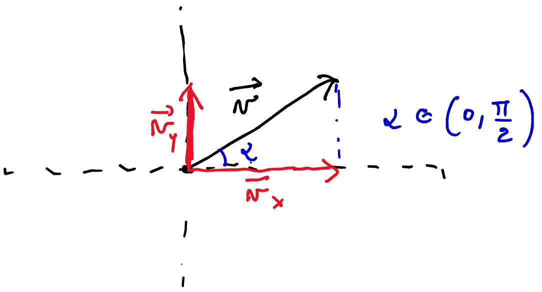
$$\cos\left(\frac{11}{8}\pi\right) = -\sqrt{\frac{1+\cos\left(\frac{11}{4}\pi\right)}{2}}$$

e $\cos\left(\frac{11}{4}\pi\right) = \cos\left(\frac{11}{4}\pi - 2\pi\right) = \cos\left(\frac{3}{4}\pi\right) = -\frac{1}{\sqrt{2}}$

$$\cos\left(\frac{11}{8}\pi\right) = -\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$

Applicazione alla fisica

Ogni vettore in $\mathbb{R}^2$ si può decomporre lungo due direzioni ortogonali.



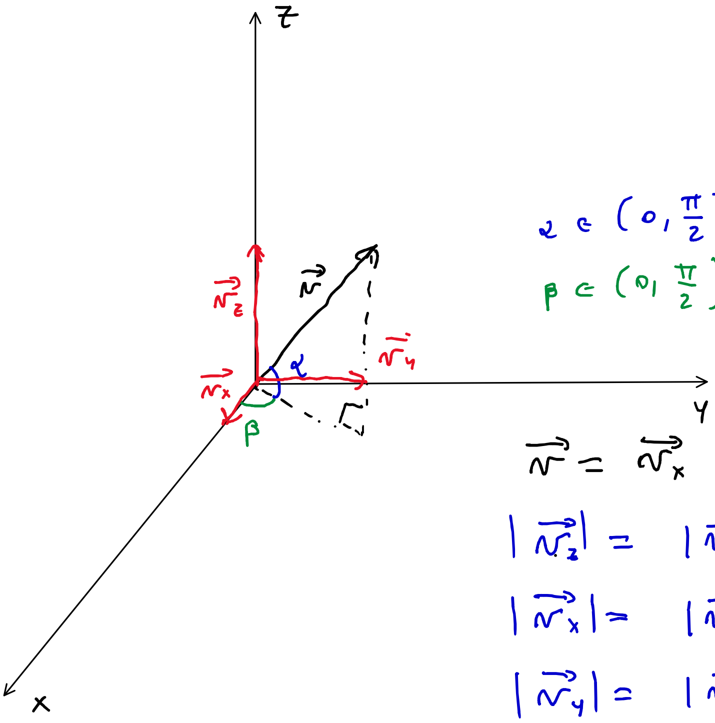
$$\vec{v} = \vec{v}_x + \vec{v}_y$$

$$|\vec{v}_x| = |\vec{v}| \cos\alpha$$

$$|\vec{v}_y| = |\vec{v}| \sin\alpha$$

- Stesso cosa per i vettori in $\mathbb{R}^3$:

si possono decomporre lungo tre direzioni ortogonali



$\alpha \in \left(0, \frac{\pi}{2}\right)$

$\beta \in \left(0, \frac{\pi}{2}\right)$

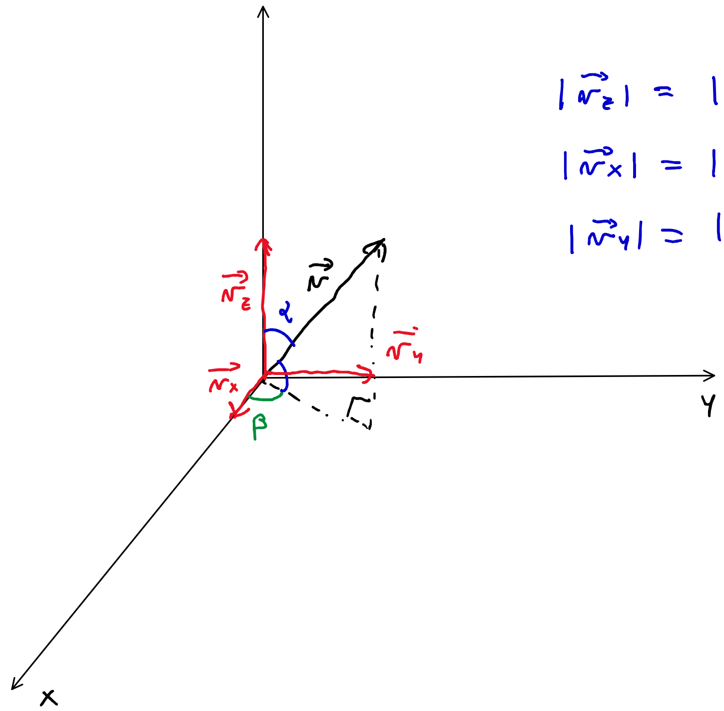
$$\vec{v} = \vec{v}_x + \vec{v}_y + \vec{v}_z$$

$$|\vec{v}_z| = |\vec{v}| \sin\alpha$$

$$|\vec{v}_x| = |\vec{v}| \cos\alpha \cos\beta$$

$$|\vec{v}_y| = |\vec{v}| \cos\alpha \sin\beta$$

Attenzione: l'espressione delle componenti dipende dalla scelta degli angoli:



$$|\vec{r}_z| = |\vec{r}| \cos\alpha$$

$$|\vec{r}_x| = |\vec{r}| \sin\alpha \cos\beta$$

$$|\vec{r}_y| = |\vec{r}| \sin\alpha \sin\beta$$